Stat 455 Cheat Sheet

Chapter 3

Conditionals

discrete pmf $P(X = x)$

continuous pdf $f_X(x) = P(x \leq X \leq x + dx)$

$E[X] = \int_{-\infty}^{\infty} x f_X(x)dx = \sum_{x \in S} xP(X = x)$

$P(A|B) = \frac{P(A \cap B)}{P(B)}$

Bayes: $P(A) = P(A|B)P(B) + P(A|B^C)P(B^C)$

conditional pmf: $p_{X|Y}(x, y) = P(X = x|Y = y) = \frac{P(X=x,Y=y)}{P(Y=y)}$

$E[X|Y = y] = \sum_x xP(X = x|Y = y) = \sum_x xp_{X|Y}(x|y)$

$E[X] = \sum_y E[X|Y = y]P(Y = y)$

$= \int_{-\infty}^{\infty} E[X|Y = y]f_Y(y)dy$

Chapter 4

Discrete Time Markov Chains

Markov Property

$P(X_{n+1} = j|X_n = i_n, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) =$

$= P(X_{n+1} = j|X_n = i_n)$

Markov Property V2

$P(X_{n+m} = j|X_n = i, \ldots, X_0 = i_0) = P(X_{n+m} = j|X_n = i)$

Transition Probabilities

$P_{ij} = P(X_{n+1} = j|X_n = i) = P(X_1 = j|X_0 = i)$

$P_{ij}^k = P(X_{n+k} = j|X_n = i) = P(X_k = j|X_0 = i)$

State J is *accessible* from I if $P_{ij}^n = (P(X_n = j|X_0 = i) > 0$,some $n > 0$

2 states *communicate* $(i \leftrightarrow j)$if they access each other

If 2 states communicate they are in the same *class.*

Any 2 classes are identical or disjoint.

A MC is *irreducible* if it has only one class.

If $N(i)$ is the number of times we visit state i, we say a state is *recurrent* if it is visited infinitely often, ie:

$P(N(i) \geq k|X_0 = i) = (f_i)^k = 1 \iff \lim_{k \to \infty} P(N(i) \geq k|X_0 = 1)$

$\Rightarrow P(N(i) = \infty|X_0 = i) = 1$

$f_i = P(\text{enter state i} \mid \text{in state i})$

If a state is not recurrent then it is *transient*, $f_i < 1, ie:$

$\lim_{k \to \infty} P(N(i) \geq k|X_0 = i) = 0 \Rightarrow P(N(i) = \infty|X_0 = i) = 0$

state i recurrent $\iff \sum_{n=1}^{\infty} P_{ii}^n = \infty$

state i transient $\iff \sum_{n=1}^{\infty} P_{ii}^n < \infty$

recurrence is a *class property*, a class property is one such that if state i has the property and $i \leftrightarrow j$ then j has the property too.

A state is *positive recurrent* if for

$\mathcal{C} = min\{n \geq 1|X_n = j\}, E[\mathcal{C}_j|X_0 = j] < \infty$

A state is *null recurrent* if $E[\mathcal{C}_j|X_0 = j] = \infty$

period of a state,d,=

$d = gcd(n \geq 1 : P_{ii}^n > 0)$

$P_{ii}^n > 0 \iff d|n$

$P_{ii}^n = 0 \iff d \nmid n$

A state is *aperiodic* if its period is 1.

A MC is *ergodic* if it is positive recurrent and aperiodic.

For an irreducible and ergodic MC $lim_{n \to \infty} P_{ij}^n = \pi_j$ is independent of i.

π_j is the solution of $\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}$

note $\sum_{j=0}^{\infty} \pi_j = 1$

π_i are the *limiting probabilities.* Put them in a vector $(\pi_1, \ldots, \pi_j)$ and that is your *stationary distribution.*

A MC is *reversible* wrt $\{\pi = i\}_{i \in S} \iff \pi_i P_{ij} = \pi_j P_{ji}$ this is the *local balance equation*

For X_n a Markov Chain, $Y_n = X_{-n}$ is the *reverse process.*

Define the *mean return time* to be $m_{jj} = inf\{n \geq 1|X_n = j\}$

$\pi_j m_{jj} = 1$ and $\pi_j = \frac{1}{\mathcal{C}_j}$

Kolmogorov Equations

$\mathbf{P}(n + m) = \mathbf{P}(n)\mathbf{P}(m)$

$P_{ij}^{n+m} = \sum_{k=0}^{\infty} P_{ik}^n P_{kj}^m$

$P_{ij}^{n+1} = P(X_{n+1} = j|X_0 = i) =$

$\sum_s P(X_{n+1} = j|X_1 = s, X_0 = i)P(X_1 = s|X_0 = i)$

Chapter 5

Exponential Distribution

$f(x) = \lambda e^{-\lambda x}, x > 0$

$F(x) = \int_{-\infty}^{x} f(x)dx = 1 - e^{-\lambda x}$

$E[X] = 1/\lambda$

$E[X^2] = 2/\lambda^2$

Memoryless RV

$P\{X > s + t|X > t\} = P\{X > s\} = e^{-\lambda s}$

Gamma Distribution

X_i drawn idd $\sim exp(\lambda) \Rightarrow X_1 + \ldots + X_n$is gamma$(n, \lambda)f(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}$

Comparing Exponentials

$P\{X_1 < X_2\} = \int_0^{\infty} P\{X_1 < X_2|X_1 = x\}\lambda_1 e^{-\lambda_1 x} dx =$

$\int_0^{\infty} P\{x < X_2\}\lambda_1 e^{-\lambda_1 x} dx = \frac{\lambda_1}{\lambda_1+\lambda_2}$

Counting Process

$\{N(t), t \geq 0\}$ is a counting process if:

- $N(t) > 0$
- $N(t)$ is integer valued
- If $s < t$ then $N(s) \leq N(t)$
- For $s < t, N(t) - N(s)$ equals the number of events that occur in the interval $(s, t]$

Independent Increments

If the numbers of events that occur in the disjoint time intervals are independent

Poisson Process

A counting process $\{N(t), t \geq 0\}$ is a Poisson process with rate λ if:

- $N(0) = 0$
- $N(t)$ has stationary and independent increments
- Number of evens in a time interval of length t is Poisson with λ: $P\{N(t + s) - N(s) = n\} = e^{-\lambda t} \frac{(\lambda t)^n)}{n!}, n = 0, 1, \ldots$
- $P(N(t) = k) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}$

$o(h)$ Functions

A function $f(\dot{)}$ is $o(h)$ if

$\lim_{h \to 0} \frac{f(h)}{h} = 0$

note that h is not $o(h)$

Poisson Process 2

- $N(0) = 0$
- The process has stationary and independent increments
- $P\{N(h) = 1\} = \lambda h + o(h)$
- $P\{N(h) \geq 2\} = o(h)$
- $P(N(h) = 0) = 1 = \lambda h + o(h)$

Poisson Process 3

- $N(0) = 0$
- $N(t)$ counts # of events that have occurred up to time t.
- Times between evens are iid $\sim exp(\lambda)$

Interarrival Time

$T_n$is the time between the $(n - 1)$st and nth events. $\{T_n\}$ is the **sequence of internarrival times** with $T_n \sim \exp(\lambda)$

$P(T_1 > t) = P(N(t) = 0) = e^{-\lambda t}$

$P(T_n > t) = e^{-\lambda t}$

Waiting Time

$S_n = \sum_{i=1}^{n} T_i \sim$gamma(n,$\lambda$)

Chapter 6

Continuous Time Markkov Chain

$\{X(t), t \geq 0\}\forall s, t \geq 0$, non-neg ints $i, j, x(u), 0 \leq u < s$ has

$P\{X(t + s) = j|X(s) = i, X(u) = x(u), 0 \leq u \leq s\}$

$= P\{X(t + s) = j|X(s) = i\}$

Stationary Transition Probabilities

A CTMC has these if $P\{X(t + s) = j|X(s) = i\}$

CTMC Alternate Definition

A stochastic process having these properties each time it enters state i:

- The amount of time it spends in that state before make a transition into a different state $\sim exp(\lambda)$ with mean $1/v_i$
- when the process leaves state i, it enters state j with some probability P_{ij} and
  $P_{ii} = 0$ all i
  $\sum_j P_{ij} = 1$, all i

Birth and Death Processes

A system with n people with

- $\{\lambda_n\}_{n=0}^{\infty}$ the arrival/birth rate
- $\{\mu_n\}_{n=0}^{\infty}$ the departure/death rate
- $v_0 = \lambda_0$
- $v_i = \lambda_i + \mu_i$
- $P_{01} = 1$
- $P_{i,i+1} = \frac{\lambda_i}{\lambda_i+\mu_i}$
- $P_{i,i-1} = \frac{\mu_i}{\lambda_i+\mu_i}$
- $q_{i,i+1} = \lambda_i$
- $q_{i,i-i} = \mu_i$
- $q_{ii} = 0$
- Knowing q's can give us P, but knowing P can't give us q's

Transition Probability Function

Move from state i to state j in a time t later.

$P_{ij} = P(X(t) = j|X(0) = i)$ a continuous function.

Instantaneous Transition Rates

$P_{ij}(t) = P\{X(t + s) = j|X(s) = i\}$

$q_{ij} = v_i P_{ij}$ is the rate, when in state i, at which the process makes a transition into state j.

$v_i = \sum_j v_i P_{ij} = \sum_j q_{ij}$

$P_{ij} = \frac{q_{ij}}{v_i} = \frac{q_{ij}}{\sum_j q_{ij}}$

T_i is the holding time in state i $\sim exp(-v_i)$

$P(T_i > h) = e^{v_i h}$

Chapman-Kolmogorov Equations

$P_{ij}(t + s) = \sum_{k=0}^{\infty} P_{ik}(t)P_{kj}(s), \forall s, t \geq 0$

Kolmogorov's Backward Equations

$P'_{ij}(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - v_i P_{ij}(t)$

Kolmogorov's Forward Equations

$P'_{ij}(t) = \sum_{k \neq j} q_{kj} P_{ik}(t) - v_j P_{ij}(t)$

Limiting Probabilities

$P_j \equiv \lim_{t\to\infty} P_{ij}(t)$ The limiting probs will exist if

- all states communicate
- the chain is positive recurrent

Use that to get:
$v_j P_i = \sum_{k\neq j} q_{kj} P_k$, all states j
(leaving = entering)
$\sum_j P_j = 1$ In the *discrete* case π may exists but limiting probabilities may not (if discrete chain is not aperiodic)
In the *continuous* case there is no similar problem, if π exists it is unique a and the above limit holds.

Embeded Chains

$\{X(t)\}_{t\geq 0}$ a CTMC
π - stationary distribution of X
ψ - stationary distribution of the embedded discrete time MC
$\psi_j = \frac{\pi_j v_j}{\sum_{i\in S} \pi_i v_i}, \pi_j = \frac{\psi_j / v_j}{\sum_{i\in S} \psi_i / v_i}$
ψ_j - long run proportoin of transitions that CTMC makes into state j
$1/v_J$ - average time it stays in state j
ψ_j / v_j - long run proportion of time the CTMC spends in state j
Note: ψ may exist and π may not!

Local Balance Equation for CTMC

$\pi_i q_{ij} = \pi_j q_{ji}$ means rate of flow from i to j = rate of flow from j to i

Time Reversability

For a long running MC, the amount of time the process spends in state i is also exponentially distributed with rate v_i. We have the discrete time reversed chain:
$Q_{ij} = \frac{\pi_j P_{ji}}{\pi_i}$ and then
$\pi_i P_{ij} = \pi_j P_{ji} \forall i, j$
$P_i q_{ij} = P_j q_{ji} \forall i, j$

Chapter 8

Definitions

L	average # of customers in system
L_Q	average # of customers waiting in queue
W	average amount of time a customer spends in system
W_Q	average amount of time a customer spends in queue
$E[S]$	average amount of time customer spends in service
$N(t)$	number of customer arrivals by time t
P_n	number of customers in system at time t $= \lim_{t\to\infty} P\{X(t) = n\}$ • aka limiting/longrun/steady state probability that n customers are in the system • also long run proportion of time that the system contains n customers
λ_a	average arrival rate of customers $= \lim_{t\to\infty} \frac{N(t)}{t}$
a_n	proportion of custs that find n in the system when arriving
d_n	proportion of custs that find n in the system when leaving

Littile's Formula

$L = \lambda_a W$ $L_Q = \lambda_a W_Q$

Poisson Model

Poisson arrivals see time averages. ie $P_n = a_n$.

M/M/1

Customers arrive according to a Poisson process with rate λ. The time between successive arrivals are independent rv with mean $1/\lambda$. If server free, cust goes in, else into queue. Service times are $\sim$exp(μ).
Balance Equations:
$\lambda P_0 = \mu P_1$
$(\lambda + \mu) P_n = \lambda P_{n-1} + \mu P_{n+1}$
$P_1 = \frac{\lambda}{\mu} P_0$
$P_{n+1} = \frac{\lambda}{\mu} P_n + (P_n - \frac{\lambda}{\mu} P_{n-1}) = \left(\frac{\lambda}{\mu}\right)^{n+1} P_0$
$1 = \sum_{n=0}^{\infty} P_n = \sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n P_0 = \frac{P_o}{1-\lambda/\mu}$ or
$\Rightarrow P_0 = 1 - \frac{\lambda}{\mu}, P_n = \left(\frac{\lambda}{\mu}\right)^n (1 - \frac{\lambda}{\mu})$
$L = \sum_{n=0}^{\infty} n P_n = \frac{\lambda}{\mu - \lambda}$
$W = \frac{L}{\lambda} = \frac{1}{\mu-\lambda}$
$W_Q = W - E[S] = W - 1/\mu = \frac{\lambda}{\mu(\mu-\lambda)}$
$L_Q = \lambda W_Q = \frac{\lambda^2}{\mu(\mu-\lambda)}$

M/M/1 - Finite Capacity

Now we have the limitation that $n \leq N$.
Balance Equations:
$\lambda P_0 = \mu P_1$
$(\lambda + \mu) P_n = \lambda P_{n-1} + \mu P_{n+1}, 1 \leq n \leq N - 1$
$\mu P_n = \lambda P_{n-1}$, for state N
$P_1 = \frac{\lambda}{\mu} P_0$
$P_{n+1} = \frac{\lambda}{\mu} P_n + (P_n - \frac{\lambda}{\mu} P_{n-1}), 1 \leq n \leq N - 1$
$P_N = \frac{\lambda}{\mu} P_{N-1} \dots = \left(\frac{\lambda}{\mu}\right)^N P_0$
$1 = \sum_{n=0}^{\infty} P_n = P_0 \sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n = P_0 \frac{1-(\lambda/\mu)^{N+1)}}{1-\lambda/\mu}$ or
$\Rightarrow P_0 = \frac{1-\lambda/\mu}{1-(\lambda/\mu)^{N+1}}, P_n = \frac{(\lambda/\mu)^n (1-\lambda/\mu)}{1-(\lambda/\mu)^{N+1}}, n = 0, \dots, N$
$L = \sum_{n=0}^{N} n P_n = \frac{(1-\lambda/\mu)}{1-(\lambda/\mu)^{N+1}} \sum_{n=0}^{N} n \left(\frac{\lambda}{\mu}\right)^n =$
$= \frac{\lambda[1+N(\lambda/\mu)^{N+1} - (N+1)(\lambda/\mu)^N]}{(\mu-\lambda)(1-(\lambda/\mu)^{N+1})}$
To find W we consider 2 cases.
$\lambda_a = \lambda$ if "customers in system" includes those who never get in
$\lambda_a = \lambda(1 - P_N)$ if it does not. Either way we get:
$W = \frac{L}{\lambda_a}$

PASTA

Poisson Arrivals See Time Averages
Let $\{X_i\}_{t\geq 0}$ a continuous-time Markov chain with stationary distribution π. Let T_i be the arrival time of the i^{th} element. These elements arrive according to a Poisson process.
Then $\{X(T_n)\}_{t\geq 0}$ has π as a stationary distribution.
$a_j = \pi_j$

Other useful stuff

$\sum_{n=0}^{\infty} n x^n = \frac{x}{(1-x)^2}$
$X \sim$ Bernoulli(p) means that you have an even with probability of success p.
$X \sim$ geometric(p) means X is the number of Bernoulli trials until success.
$p(n) = p\{X = n\} = (1 - o)^{n-1} p$
$X \sim$ binomial(n, p) X is the number of successes in n trials.
$p(i) = \binom{n}{i} p^i (1 - p)^{n-i}$
$\binom{n}{i} = \frac{n!}{(n-i)!i!}$
$\sum_{i=0}^{N} \binom{N}{i} = (1 + 1)^N = 2^N$
$\sum_{i=1}^{k} m^i = \frac{m(1-m^k)}{1-m}$

Infitesimal Generator: G

$g_{ij} = q_{ij}, i \neq j$
$g_{ii} = -v_i$ (note: on last 2 lines, entries not probabilities
$[P\prime(t)]_{ij} = [GP(t)]_{ij}$
$P\prime(t) = GP(t)$(K's backwards eqn)
$P\prime(t) = P(t)G$ (K's forward eqn)
$P(t) = e^{tg} = \sum_{n=0}^{\infty} \frac{(tG)^n}{n!}$
a

Stationary Distribution for a CTMC

$\pi = \pi P(t)$
$\sum \pi = 1$
If the initial distribution of $X(0)$ is π then the distribution of $X(t)$ will also be $\pi, t > 0$. $(P(X(t) = j) = \pi_j \forall j \in S, t > 0$
$\sum_{i\in S} p_{ij}(t) \pi_i = \pi_j$

Global balance equation for CTMC

$0 = \pi G$
$v_j \pi_j = \sum_{i\neq j} q_{ij} \pi_i$ this is the jth row of the matrix
$\Rightarrow$ long run rate out of j = long run rate into state i

π_i long run proportion of time that process in in state j.
v_j rate of leaving state j
$\pi_j v_j$ long run rate of leaving j
$\pi_i q_{ij}$ long run rate of going from state i to j